MA 335, Spring 2012

Assignment 4.

This homework is due *Thursday* Feb 16.

There are total 55 points in this assignment. 50 points is considered 100%. If you go over 50 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

(1) Determine which of the following Diophantine equations can be solved:

  (a) [2pt] (2.5.1c) $14x + 35y = 93$,
  (b) [2pt] (2.5.1b) $33x + 14y = 115$,
  (c) [3pt] $33x + 57y - 9z = 10$.

(2) (2.5.2ac) Find all solutions in the integers of the equation (use reverse Euclidean algorithm)
  (a) [4pt] $56x + 72y = 40$.
  (b) [4pt] $221x + 35y = 11$.

(3) [4pt] (2.5.3b) Determine all solutions in the *positive* integers of the equation (use reverse Euclidean algorithm) $54x + 21y = 906$.

(4) (a) [4pt] Show that if $p_1, p_2, \dots p_n$ are any distinct primes, then
$$\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_n}$$
is not an integer. (For example: $1/2 + 1/3 + 1/6 = 1 \in \mathbb{Z}$, but 6 is not a prime.)
  (b) [0pt, just an interesting problem] Show that if $n > 1$, then
$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$
is never an integer.

(5) [3pt] Find all triples of primes of the form $p, p + 2, p + 4$.

(6) [4pt] (3.3.13) Prove that there are infinitely many primes of the form $6n+5$.

(7) [3pt] Prove that if n divides $(n - 1)! + 1$, then n is prime. (*Hint:* See Problem 11 in HW3.)

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(8) (a) [3pt] (4.2.1a) Find the remainders when 2^{50} and 41^{65} are divided by 7.

(b) [3pt] ($\sim$4.2.1b) What is the remainder when the following sum is divided by 4?

$$1^5 + 2^5 + 3^5 + \ldots + 2011^5 + 2012^5$$

(c) [3pt] (part of 4.1.5) Prove that $53^{103} + 103^{53}$ is divisible by 39.
(*Hint:* $39 = 13 \cdot 3$.)

(d) [3pt] (part of 4.1.5) Prove that $111^{333} + 333^{111}$ is divisible by 7.

(9) (a) [3pt] Prove that if p is a prime and $1 \le k \le p-1$, then $p \mid \binom{p}{k}$.

(b) [2pt] Prove that $(a+b)^p \equiv a^p + b^p \pmod{p}$ for any integers a, b.

(c) [2pt] Prove that $(a_1 + a_2 + \ldots + a_n)^p = a_1^p + a_2^p + \ldots + a_n^p \pmod{p}$ for any integers $a_1, \ldots, a_n$.
(*Hint:* $a_1 + a_2 + \ldots + a_n = (a_1 + \ldots + a_{n-1}) + a_n$. Use induction.)

(d) [3pt] Prove that for any integer a one has $a^p \equiv a \pmod{p}$.
(*Hint:* If $a > 0$, then $a = 1 + 1 + \ldots + 1$.)

COMMENT. Congratulations, you just proved Fermat's Little Theorem. I will give a different proof in class at some point.